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Dwork-Carlitz Exponential and Overconvergence for Additive Functions in Positive Characteristic

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Abstract

We study overconvergence phenomena for $\mathbb{F}_q$-linear functions on a function field over a finite field $\mathbb{F}_q$. In particular, an analog of the Dwork exponential is introduced.

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1. One of the central subjects of contemporary p-adic analysis is that of overconvergence. In contrast to analysis over $\mathbb{R}$ and $\mathbb{C}$, the power series for principal special functions over $\mathbb{Q}_p$ or $\mathbb{C}_p$ converge only on finite disks or annuli. For example, the exponential series $\exp(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, $t \in \mathbb{C}_p$, converges if and only if $|t|_p < p^{-1/(p-1)}$ (see [6] or [15]).

At the same time, for many special functions there exist some expressions combining their values in various points (usually connected by the Frobenius power $t \mapsto t^p$), for which the corresponding power series converge on wider regions. The simplest example is the Dwork exponential

$$\theta(t) = \exp(\pi(t - t^p)) \tag{1}$$

where π is a root of the equation $z^{p-1} + p = 0$. The power series for $\theta(t)$, in the variable t, converges for $|t|_p < p^{\frac{p-1}{p^2}}$ (> 1), though the formula (1) is not valid outside the unit disk. The special value $\theta(1)$ is a primitive p-th root of unity.

Other examples involve the exponential function of q-analysis [1], some hypergeometric functions [5], polylogarithms [3], and many others. The overconvergent functions usually satisfy equations possessing special algebraic properties called the Frobenius structures (see [1, 14]).

In this paper we consider the overconvergence phenomena in the case of a local field of a positive characteristic, that is (up to an isomorphism) the field K of formal Laurent series

$$z = \sum_{i=n}^{\infty} \zeta_i x^i, \quad n \in \mathbb{Z},\ \zeta_i \in \mathbb{F}_q, \tag{2}$$

with coefficients from a finite field $\mathbb{F}_q$. If $z \in K$ is an element (2) with $\zeta_n \neq 0$, its non-Archimedean absolute value $|z|$ is given by $|z| = q^{-n}$. Note that the above construction of K (in contrast to the completion of $\mathbb{F}_q(x)$ with respect to the ∞-valuation often used in the function field arithmetic) leads to structures (operators, orthonormal bases etc) resembling to some extent the p-adic case; see, for example, [9, 4].

We will consider only $\mathbb{F}_q$-linear functions and power series of the form $\sum_{k=0}^{\infty} c_k t^{q^k}$, $c_k \in \overline{K}_c$ ($\overline{K}_c$ is the completion of an algebraic closure $\overline{K}$ of K). This class contains many important functions, including the Carlitz exponential and logarithm, analogs of the Bessel and hypergeometric functions, polylogarithms etc.

In particular, using the Carlitz exponential e_C (see below), we construct an analog of the Dwork exponential and prove its overconvergence, consider the overconvergence problems for other special functions mentioned above. These problems are much simpler than those in the characteristic zero case. The reason is that the above functions satisfy differential equations with the Carlitz derivatives (see [10, 11, 12, 13, 17]); the difference structure of the latter leads immediately to overconvergence properties of some linear combinations of solutions.

2. The Carlitz exponential (see [8, 17]) is the function

$$e_C(t) = \sum_{n=0}^{\infty} \frac{t^{q^n}}{D_n} \tag{3}$$

where D_n is the Carlitz factorial

$$D_n = [n][n-1]^q \dots [1]^{q^{n-1}}, \quad [n] = x^{q^n} - x \ (n \geq 1),\ D_0 = 1. \tag{4}$$

Since $|[n]| = q^{-1}$ for any $n \geq 1$, it follows from (4) that

$$|D_n| = q^{-\frac{q^n-1}{q-1}},$$

so that the series in (3) converges for $|t| < q^{-\frac{1}{q-1}}$.

Let σ be an arbitrary solution of the equation $z^{q-1} = -x$. Let us consider the function

$$E(t) = e_C(\sigma(t - t^q)), \tag{5}$$

defined initially for $|t| < q^{-\frac{1}{q-1}}$ (we denote by $|\cdot|$ also the extension of the absolute value from K onto $\overline{K}_c$). Note that, in spite of a formal resemblance, the formulas for the Dwork exponential (1) and "the Dwork-Carlitz exponential" (5) have a quite different meaning – the function $\theta(t)$ is a multiplicative combination of values of the classical exponential, while $E(t)$ is an additive combination of values of the Carlitz exponential. This difference from the classical overconvergence theory appears also in some other examples given below.

From (3) and (5), after a simple transformation we find that

$$E(t) = \sigma t + \sum_{n=1}^{\infty} \left(\frac{\sigma^{q^n}}{D_n} - \frac{\sigma^{q^{n-1}}}{D_{n-1}} \right) t^{q^n}. \tag{6}$$

In order to investigate the convergence of the series (6), we have to study the structure of elements D_n.

Proposition 1. *For any $n \geq 1$,*

$$\left| D_n - (-1)^n x^{1+q+\cdots+q^{n-1}} \right| \leq q^{-\frac{q^n-1}{q-1}-(q-1)q^{n-1}}. \tag{7}$$

Proof. We will prove that

$$\left| D_n - (-1)^n x^{1+q+\cdots+q^{n-1}} \right| \leq q^{-l_n} \tag{8}$$

where the sequence $\{l_n\}$ is determined by the requrrence

$$l_n = ql_{n-1} + 1, \quad l_1 = q. \tag{9}$$

Indeed, if $n = 1$, then $D_1 = x^q - x$, so that $|D_1 + x| = q^{-q}$. Suppose that we have proved (8) for some value of n. We have

$$\left| D_n^q - (-1)^n x^{q+q^2\cdots+q^n} \right| \leq q^{-ql_n},$$

whence

$$\left| [n+1]D_n^q - (-1)^n [n+1] x^{q+q^2\cdots+q^n} \right| \leq q^{-(ql_n+1)}.$$

Since $D_{n+1} = [n+1]D_n^q$, we find that

$$\left| \left(D_{n+1} - (-1)^{n+1} x^{1+q+\cdots+q^n} \right) - (-1)^n x^{q+q^2\cdots+q^{n+1}} \right| \leq q^{-l_{n+1}}. \tag{10}$$

It is easy to check that

$$l_n = \frac{q^n - 1}{q - 1} + (q-1)q^{n-1} \tag{11}$$

satisfies (9); the expression (11) can also be deduced from a general formula for a solution of a difference equation; see [7].

On the other hand,

$$\left|(-1)^n x^{q+q^2\cdots+q^{n+1}}\right| = q^{-\frac{q^{n+2}-q}{q-1}}, \tag{12}$$

and, by a simple computation,

$$\frac{q^{n+2}-q}{q-1} - l_{n+1} = q^n - 1 > 0, \quad n \geq 1. \tag{13}$$

It follows from (10), (12), (13), and the ultra-metric property of the absolute value, that

$$\left|D_{n+1} - (-1)^{n+1} x^{1+q+\cdots+q^n}\right| \leq q^{-l_{n+1}},$$

which proves the inequalities (8) and (7) for any n. ■

Now we can prove the overconvergence of $E(t)$.

Proposition 2. *The series in (6) converges for* $|t| < \rho$, *where* $\rho = q^{\frac{q-1}{q^2}} > 1$. *In particular,* $E(1) = \lim\limits_{n\to\infty} \frac{\sigma^{q^n}}{D_n}$ *is defined, and* $E(1) = \sigma$.

Proof. Let us write

$$\frac{\sigma^{q^n}}{D_n} - \frac{\sigma^{q^{n-1}}}{D_{n-1}} = \frac{\sigma^{q^{n-1}}}{D_{n-1}} \left(\frac{\sigma^{q^n - q^{n-1}} D_{n-1}}{D_n} - 1 \right).$$

We have $\sigma^{q^n-q^{n-1}} = -x^{q^{n-1}}$, so that

$$\begin{aligned} \frac{\sigma^{q^n}}{D_n} - \frac{\sigma^{q^{n-1}}}{D_{n-1}} &= -\frac{\sigma^{q^{n-1}}}{D_{n-1}D_n} \left(x^{q^{n-1}} D_{n-1} + D_n \right) \\ &= -\frac{\sigma^{q^{n-1}}}{D_{n-1}D_n} \left\{ x^{q^{n-1}} \left(D_{n-1} - (-1)^{n-1} x^{1+\cdots+q^{n-2}} \right) + \left(D_n - (-1)^n x^{1+\cdots+q^{n-1}} \right) \right\}. \end{aligned}$$

If $n \geq 2$, then by Proposition 1,

$$\left| x^{q^{n-1}} \left(D_{n-1} - (-1)^{n-1} x^{1+\cdots+q^{n-2}} \right) \right| \leq q^{-\left(q^{n-1} + \frac{q^{n-1}-1}{q-1} + (q-1)q^{n-2} \right)},$$

$$\left| D_n - (-1)^n x^{1+\cdots+q^{n-1}} \right| \leq q^{-\frac{q^n-1}{q-1} - (q-1)q^{n-1}}.$$

Comparing the right-hand sides we check that the first of them is bigger; therefore

$$\left| \frac{\sigma^{q^n}}{D_n} - \frac{\sigma^{q^{n-1}}}{D_{n-1}} \right| \leq q^{-\frac{q^{n-1}}{q-1}} \cdot q^{-\frac{q^{n-1}-1}{q-1}} \cdot q^{-\frac{q^n-1}{q-1}} \cdot q^{-\left(q^{n-1} + \frac{q^{n-1}-1}{q-1} + (q-1)q^{n-2} \right)},$$

so that

$$\left|\frac{\sigma^{q^n}}{D_n}-\frac{\sigma^{q^{n-1}}}{D_{n-1}}\right|\le q^{-\frac{q^{n-2}(q-1)^2+1}{q-1}},\quad n\ge 2. \tag{14}$$

For $n=1$, we get

$$\left|\frac{\sigma^q}{D_1}-\sigma\right|=|\sigma|\left|\frac{-x}{[1]}-1\right|=|\sigma|\left|\frac{x^q}{[1]}\right|,$$

whence

$$\left|\frac{\sigma^q}{D_1}-\sigma\right|\le q^{-\frac{1}{q-1}-(q-1)}. \tag{15}$$

It follows from (14) that the series in (6) converges for $|t|<\rho$. For $t=1$, we obtain that

$$E(1)=\sigma+\sum_{n=1}^{\infty}\left(\frac{\sigma^{q^n}}{D_n}-\frac{\sigma^{q^{n-1}}}{D_{n-1}}\right)=\lim_{n\to\infty}\frac{\sigma^{q^n}}{D_n}. \tag{16}$$

Note that

$$\left|\frac{\sigma^{q^n}}{D_n}\right|=q^{-\frac{1}{q-1}}$$

for all values of n. Now

$$E(1)^q=\lim_{n\to\infty}\frac{\sigma^{q^{n+1}}}{D_n^q}=\lim_{n\to\infty}[n+1]\frac{\sigma^{q^{n+1}}}{D_{n+1}}=\lim_{n\to\infty}[n]\frac{\sigma^{q^n}}{D_n}=-xE(1)$$

because

$$\left|\frac{x^{q^n}\sigma^{q^n}}{D_n}\right|=q^{-\frac{1}{q-1}-q^n}\longrightarrow 0,$$

as $n\to\infty$.

By (16), $E(1)\ne 0$, so that $E(1)^{q-1}=-x$, thus $E(1)$ satisfies the same equation as σ. All the solutions of this equation are obtained by multiplying σ by non-zero elements $\xi\in\mathbb{F}_q$. Therefore $E(1)=\sigma\xi$, $\xi\in\mathbb{F}_q$, $\xi\ne 0$. If $\xi\ne 1$, then

$$|E(1)-\sigma|=|(1-\xi)\sigma|=|\sigma|=q^{-\frac{1}{q-1}}. \tag{17}$$

On the other hand, by (16),

$$|E(1)-\sigma|\le\sup_{n\ge 1}\left|\frac{\sigma^{q^n}}{D_n}-\frac{\sigma^{q^{n-1}}}{D_{n-1}}\right|,$$

and we see that (17) contradicts (14) and (15). ■

It is interesting that the special value $\sigma=E(1)$, just as the special value of the Dwork exponential in the characteristic 0 case, generates a cyclotomic extension of the function field (related in this case to the Carlitz module); see [16].

3. As it has been mentioned, many important $\mathbb{F}_q$-linear functions defined on subsets of K satisfy equations involving the Carlitz derivative $d=\sqrt[q]{\ }\circ\Delta$ where

$$\Delta u(t)=u(xt)-xu(t).$$

The Carlitz exponential e_C satisfies the simplest equation $de_C = e_C$, so that

$$e_C(t)^q + xe_C(t) = e_C(xt). \tag{18}$$

The right-hand side of (18) obviously converges on a wider disk than e_C itself (note that, in contrast to the p-adic case, $E(t)$ does not satisfy a homogeneous equation with the Carlitz derivative).

Similarly, the Bessel-Carlitz function $J_n(t)$, introduced in [2], satisfies the identity $\Delta J_n = J_{n-1}^q$, so that

$$J_{n-1}^q(t) + xJ_n(t) = J_n(xt), \tag{19}$$

and we have an overconvergence for the right-hand side of (19). In this sense equations with the Carlitz derivatives may be seen themselves as analogs of the Frobenius structures of p-adic analysis.

The next two examples (of an essentially similar nature) are just a little more complicated.

4. Polylogarithms on K, in the sense of [12], are defined as follows. First the function $l_1(t)$, an analog of the function $-\log(1-t)$, is introduced as a solution of the equation $(1-\tau)du(t) = t$, where $\tau u = u^q$. This equation is, of course, an analog of the classical equation $(1-t)u'(t) = 1$ (note that the function $f(t) = t$ is the unit element in the composition rings of $\mathbb{F}_q$-linear polynomials or holomorphic functions). Then the polylogarithms $l_n(t)$ are defined reqursively by the equations $\Delta l_n = l_{n-1}$, $n \geq 2$ (classically, $tl_n'(t) = l_{n-1}(t)$). These definitions lead [12] to the explicit expressions

$$l_n(t) = \sum_{j=1}^{\infty} \frac{t^{q^j}}{[j]^n}, \quad n = 1, 2, \dots. \tag{20}$$

The series in (20) converges for $|t| < 1$. In [12] we constructed their continuous extensions to the disk $\{t \in K : \ |t| \leq 1\}$. Here we give the following overconvergence result resembling Coleman's theorem [3] about classical polylogarithms.

Proposition 3. *The power series for the function $L_n(t) = l_n(t) - l_n(t^q)$ converges for $|t| < q^{1/q}$.*

Proof. By a simple transformation, we get

$$L_n(t) = \frac{t^q}{[1]^n} + \sum_{j=2}^{\infty} \left(\frac{1}{[j]^n} - \frac{1}{[j-1]^n} \right) t^{q^j}. \tag{21}$$

We have,

$$\frac{1}{[j]^n} - \frac{1}{[j-1]^n} = \frac{([j-1]-[j])\left([j]^{n-1} + [j-1][j]^{n-2} + \cdots + [j-1]^{n-1}\right)}{[j]^n[j-1]^n}.$$

For any $j \geq 2$, $|[j]| = q^{-1}$, $|[j-1]-[j]| = \left| x^{q^{j-1}} - x^{q^j} \right| = q^{-q^{j-1}}$, so that

$$\left| \frac{1}{[j]^n} - \frac{1}{[j-1]^n} \right| \leq q^{-q^{j-1}+n-1},$$

and the convergence radius of the series (21) equals $q^{1/q}$. ■

5. Let us consider the hypergeometric function [13]

$$F(a,b;c;t)=\sum_{n=0}^{\infty}\frac{\langle a\rangle_n\langle b\rangle_n}{\langle c\rangle_n D_n}t^{q^n}, \tag{22}$$

where $a,b,c\in\overline{K}_c$, $c\notin\{[0],[1],\dots,[\infty]\}$, $[\infty]=-x$, and the Pochhammer-type symbols are defined as $\langle a\rangle_0=1$,

$$\langle a\rangle_m=([0]-a)^{q^n}([1]-a)^{q^{n-1}}\cdots([n-1]-a)^q,\quad n\ge 1.$$

If all the parameters have the form $[-\alpha]$, $\alpha\in\mathbb{Z}$, then the function (22) coincides, up to a change of variable, with the hypergeometric function introduced by Thakur [17].

Denote $T_1(a)=(a-[1])^{1/q}$, $a\in\overline{K}_c$. The transformation T_1 is an analog of the unit shift of integers: if $a=[-\alpha],\alpha\in\mathbb{Z}$, then $T_1([-\alpha])=[-\alpha-1]$. The identity

$$\langle a\rangle_n=-a^{q^n}\langle T_1(a)\rangle_{n-1}^q,\quad n\ge 1, \tag{23}$$

holds for any $a\in\overline{K}_c$ (see [13]).

If $|a|=|b|=|c|=1$, then $|T_1(a)|=|T_1(b)|=|T_1(c)|=1$, and the disk of convergence of the series (22) is the same as the one for the Carlitz exponential, that is $\left\{t\in\overline{K}_c:\ |t|<q^{-\frac{1}{q-1}}\right\}$.

Proposition 4. *The identity*

$$\tau F(T_1(a),T_1(b);T_1(c);\frac{ab}{c}t)-xF(a,b;c;t)=-F(a,b;c;xt) \tag{24}$$

holds for any values of the variable and parameters, such that all the terms of (24) make sense. In particular, if $|a|=|b|=|c|=1$, then the right-hand side of (24) is overconvergent, that is the series for the right-hand side converges for $|t|<q^{1-\frac{1}{q-1}}$ (>1).

Proof. Changing the index of summation we find that

$$\tau F(T_1(a),T_1(b);T_1(c);z)=\sum_{n=0}^{\infty}\frac{\langle T_1(a)\rangle_{n-1}^q\langle T_1(b)\rangle_{n-1}^q}{\langle T_1(c)\rangle_{n-1}^q D_{n-1}^q}t^{q^n}$$

for any z from the convergence disk. Using the identity (23) and the fact that $D_n=[n]D_{n-1}^q$ we get

$$\tau F(T_1(a),T_1(b);T_1(c);z)=-\sum_{n=0}^{\infty}\frac{\langle a\rangle_n\langle b\rangle_n[n]}{\langle c\rangle_n D_n}\left(\frac{ab}{c}\right)^{-q^n}z^{q^n}$$

(note that $[0]=0$), which implies (24). ■

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